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Upscaling from particle models to entropic gradient flows

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We prove that, for the case of Gaussians on the real line, the functional derived by a time discretization of the diffusion equation as entropic gradient flow is asymptotically equivalent to the rate functional derived from the underlying microscopic process. This result strengthens a conjecture that the same statement is actually true for all measures with second finite moment. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4726509>]

I. INTRODUCTION

It is well known that the diffusion equation

$$\frac{\partial \rho}{\partial t}(x) = \Delta \rho(x), \quad t \in [0, \infty), \quad x \in \mathbb{R}^n$$

can be interpreted as a gradient flow in several ways. For example, it is the gradient flow of the Dirichlet integral $\frac{1}{2} \int |\nabla \rho|^2 dx$ with respect to the L^2 metric, and the gradient flow of $\frac{1}{2} \int \rho^2 dx$ with respect to the H^{-1} metric. A different and physically natural formulation, describing diffusive evolution as gradient flow of the entropy functional, was introduced in Ref. 5 (with respect to the Wasserstein distance on the space of probability measures with finite second moment $\mathcal{P}_2(\mathbb{R}^n)$, see below). This article provides a link between a microscopic model and this entropic formulation of the diffusion equation, for the case of Gaussian measures on the real line.

This link seems to be natural: it is a classic fact that the diffusion equation is the macroscopic limit of a Brownian particle system, in the sense that the empirical distribution of the latter converges to a solution of the diffusion equation when the number of particles tends to ∞ . One would thus expect that the macroscopic behaviour of the Brownian particles will be governed by entropy, and one key achievement of Jordan, Kinderlehrer, and Otto is to make this macroscopic entropic nature apparent.

We now describe the setting more precisely. The *entropy* is defined as $E(P) = \int \rho \log(\rho) dx$, where ρ is the Lebesgue density of P when the measure is absolutely continuous with respect to Lebesgue measure and ∞ otherwise. The 2-Wasserstein distance of two measures $P', P \in \mathcal{P}_2(\mathbb{R}^n)$ is defined via

$$W_2^2(P', P) = \inf_{Q \in \Pi(P', P)} \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|x - y\|_2^2 dQ \right\},$$

where $\Pi(P', P)$ is the set of all Borel measures in \mathbb{R}^{2n} that have first and second marginal P' and P , respectively.

In Ref. 5, a time-discrete scheme is considered, giving rise to the entropic gradient flow. For $h > 0$, one defines recursively a sequence of measures $\{P^n\}$ by

$$P^n \in \arg \min_{P \in \mathcal{P}_2(\mathbb{R})} K_h(P; P^{n-1}), \quad (1)$$

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where

$$K_h(P; P^{n-1}) := \frac{1}{2h} W_2^2(P, P^{n-1})^2 + E(P) - E(P^{n-1}). \quad (2)$$

It is proved in Ref. 5 that a linear interpolation of the points obtained in this way yields a sequence of functions $P_h(t)$ that converges to the solution of the diffusion equation as $h \rightarrow 0$.

Our goal is to highlight a new connection between the functional capturing the large deviations of the microscopic model and functional K_h in (2). The main result of this paper is that the functional K_h is asymptotically equivalent to the rate functional obtained from a particle model for diffusion, for all Gaussian measures $\mathcal{N}(\mu, \sigma^2)$, where μ is the mean, σ^2 the variance and the density with respect to Lebesgue measure is given by

$$v(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

For a different class of measures, namely relatively uniform measures on a flat torus, this result has been proved by Adams, Dirr, Peletier, and Zimmer.¹

Jordan, Kinderlehrer, and Otto considered the time discretization scheme (2), due to the lack of a rigorously defined differential structure on the space of probability measures at that time. This problem was then addressed subsequently. In an exceptional paper,⁸ Otto made the first step towards treating $\mathcal{P}_2(\mathbb{R}^n)$ as an infinite dimensional Riemannian-like manifold, by introducing some differential structure and establishing what is known today as “Otto’s calculus” (this construction, although it equips every point in $\mathcal{P}_2(\mathbb{R}^n)$ with a tangent cone and the whole space with a Riemannian metric, gives rise to an exponential map which is not a bijection). This programme was completed by Ambrosio, Gigli, and Savaré in Ref. 2 by proving that every two points in $\mathcal{P}_2(\mathbb{R}^n)$ can be connected by an absolutely continuous geodesic and that every absolutely continuous curve has a derivative that lives inside the respective tangent cone almost everywhere and it can be reconstructed by those derivatives via the mass preservation equation.

It is remarkable that the Wasserstein structure and entropic gradient flows appear earlier in the probability community. Dawson and Gärtner^{3,4} studied Brownian particles under the influence of an on-site potential. They obtain a Large Deviation principle for paths of probability measures, with the rate functional given by

$$\int_0^T \left\| \frac{\partial \rho_t}{\partial t} - \Delta \rho_t + \operatorname{div}(\nabla \Psi \cdot \rho_t) \right\|_{-1, \rho_t}^2 dt, \quad (3)$$

where the norm is the same as the one used by Otto. Dawson and Gärtner also noticed that in some cases, the evolution can be interpreted as the Wasserstein gradient flow of the free energy [Ref. 4, Eq. (1.10)].

From the large deviation results of now classic papers,^{3,4,6} the time-discrete one used here can be derived, e.g., using the contraction principle. Since the equivalence between the diffusion equation and the entropic gradient flow is known, a natural connection between Brownian particles and entropic gradient flows can be derived in this way. However, we pursue the different approach of showing the asymptotic equivalence of the rate functional of the microscopic process and the time discretization of entropic gradient flow for several reasons: the argument is elementary and does not use the deep (and in the time-continuous case highly nontrivial) equivalence between diffusion and entropic gradient flows. Also, an advantage of the time-discrete approach taken here is that entropy as driving force is derived directly in a natural way. Finally, as described in Ref. 1, a consequence of this asymptotic equivalence is that the functional K_h not only gives a way to approximate the solution of the diffusion equation, but also characterises the fluctuation behaviour of the system.

II. LARGE DEVIATION PRINCIPLE

Let S be a Polish space, i.e., a complete separable metric space, and $\{X_n\}_{n \in \mathbb{N}}$ a sequence of random variables with values in S . A large deviations principle (LDP) assigns to such a sequence a non-negative function $I(x)$ which asymptotically grades the rarity of the observables.

Definition: Let S be a Polish space and $\{X_n\}_{n \in \mathbb{N}}$ a sequence of random variables on S . Let I be a lower semi-continuous function from S to $[0, +\infty]$. We say that X_n satisfies a *large deviation principle* with *rate function* I if

1. For each closed subset F of S ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \in F) \leq - \inf_{x \in F} I(x).$$

2. For each open subset G of S ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \in G) \geq - \inf_{x \in G} I(x).$$

Obviously, by definition, for a system satisfying a LDP, the probability of observing something unexpected decays exponentially after some point.

Let us consider a collection of Brownian particles initially situated at points x_1, x_2, \dots , where x_i , with $i \in \mathbb{N}$, satisfy

$$\sum_{i=1}^n \delta_{x_i} \rightharpoonup P_0,$$

for some measure $P_0 \in \mathcal{P}_2(\mathbb{R})$, with the topology of $\mathcal{P}_2(\mathbb{R})$ being the narrow topology. Each particle is moving independently from the others and their positions after some time $h > 0$ is given by the random variables $Y_i(h) = x_i + U_i(h)$, where $U_i(h)$ are independent copies of $U(h)$ with law $U(h) = \mathcal{N}(0, 2h)$, the normal distribution with mean 0 and variance $2h$. For each $n \in \mathbb{N}$, the law that governs the evolution of the n first particles is $\mathbb{P}_n = \otimes_{i=1}^n \mathbb{P}_{x_i}$. It is expected from the law of large numbers that as n becomes larger, $L_n(h)$ converges in probability to $P(h) = P_0 * \mathcal{N}(0, 2h)$, where the symbol $*$ stands for convolution of measures. In Ref. 7, it is proven that the sequence

$$L_n(h) = \frac{1}{n} \sum_{i=1}^n \delta_{Y_i(h)}$$

satisfies a LDP with rate function

$$J_h(P; P_0) = \inf_{Q \in \Pi(P_0, P)} H(Q \| Q_{0 \rightarrow h}) = \inf_{Q_x: P_0 * Q_x = P} \int H(Q_x \| P_{h,x}) dP_0(x), \quad (4)$$

with

$$H(Q' \| Q) = \int_{\mathbb{R}^n} \frac{dQ'}{dQ} \log\left(\frac{dQ'}{dQ}\right) dQ = \int_{\mathbb{R}^n} \log\left(\frac{dQ'}{dQ}\right) dQ'$$

being the *relative entropy* of Q' with respect to Q ; here the *diffusion kernel* $P_{h,x}$ is absolutely continuous for every x and its Lebesgue density is given by

$$p_{h,x}(y) = p_h(x; y) = \frac{1}{\sqrt{4\pi h}} e^{-\frac{(x-y)^2}{4h}}.$$

III. THE CENTRAL STATEMENT

The aim of this paper, as of its predecessor,¹ is to connect J_h to the functional K_h in the limit $h \rightarrow 0$, in the sense that

$$J_h(\cdot; P_0) \sim \frac{1}{2} K_h(\cdot; P_0) \quad \text{as } h \rightarrow 0.$$

For any $P \neq P_0$, both $J_h(P; P_0)$ and $K_h(P; P_0)$ diverge as $h \rightarrow 0$; however, and we therefore formulate this statement in the form

$$J_h(\cdot; P_0) - \frac{1}{4h} W_2^2(\cdot, P_0) \rightarrow \frac{1}{2} E(\cdot) - \frac{1}{2} E(P_0).$$

The convergence of functionals is to be understood in the sense of Γ -convergence, and we recall the definition for the reader's convenience.

Definition: Let S be a topological space that satisfies the first axiom of countability. We say that F_n Γ -converges to F if the following conditions are satisfied:

1. For every $x \in S$ and for every sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to x in S ,

$$F(x) \leq \liminf_{n \rightarrow \infty} F_n(x_n).$$

2. For every $x \in S$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to x in S such that

$$F(x) = \lim_{n \rightarrow \infty} F_n(x_n).$$

We then write $F_n \xrightarrow{\Gamma} F$.

We now state the main result of this article.

Theorem 3.1: Let $\mathcal{N}(\mu_0, \sigma_0^2)$ be a normal distribution. Then for the rate functional J_h it holds that

$$J_h(\cdot; \mathcal{N}(\mu_0, \sigma_0^2)) - \frac{W_2^2(\cdot, \mathcal{N}(\mu_0, \sigma_0^2))}{4h} \rightarrow \frac{1}{2} E(\cdot) - \frac{1}{2} E(\mathcal{N}(\mu_0, \sigma_0^2)),$$

locally uniform and in the sense of Γ -convergence with respect to the weak topology, on the submanifold of the Gaussians.

The proof of the central statement is as follows. First we explicitly calculate the minimizer of $H(\cdot \| Q_{0 \rightarrow h})$ for every h, P , and P_0 . Then we estimate the difference with the functional $\frac{1}{2} K_h(P_0; P)$; the statement then follows easily due from the topological structure of the Gaussian submanifold.

IV. MINIMIZING THE RELATIVE ENTROPY OVER A BIVARIATE

For clarity, we use a boldface font for vectors in this section and in Sec. V.

Theorem 4.1: Let Q be absolutely continuous with respect to the Lebesgue measure in \mathbb{R}^n . Furthermore, let \mathcal{Q}' be the set of all Borel measures Q' in \mathbb{R}^n satisfying

$$\int r_i(\mathbf{x}) \cdot q'(\mathbf{x}) d\mathbf{x} = a_i, \quad i \in \{1, 2, \dots, N\}, \quad (5)$$

where $r_i(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ are given functions and $a_i \in \mathbb{R}$. Let us finally assume that there is a measure $Q^* \in \mathcal{Q}'$ that has a density of the form

$$q^*(\mathbf{x}) = q(\mathbf{x}) \exp \left(\sum_{i=1}^n \lambda_i r_i(\mathbf{x}) \right)$$

for some $\lambda_i \in \mathbb{R}$. Then Q^* is the unique minimizer of $H(Q' \| Q)$ over all Q' in \mathcal{Q}' .

Proof: We can assume that $Q' \ll Q$ since otherwise we trivially have that $H(Q' \| Q) = \infty$. Due to the form of Q^* it is also true that $Q' \ll Q^*$. Now

$$\begin{aligned} H(Q' \| Q) &= \int q'(\mathbf{x}) \log \frac{q'(\mathbf{x})}{q(\mathbf{x})} d\mathbf{x} = \int q'(\mathbf{x}) \log \frac{q'(\mathbf{x})}{q^*(\mathbf{x})} d\mathbf{x} + \int q'(\mathbf{x}) \log \frac{q^*(\mathbf{x})}{q(\mathbf{x})} d\mathbf{x} \\ &= H(Q' \| Q^*) + \int q'(\mathbf{x}) \log \frac{q^*(\mathbf{x})}{q(\mathbf{x})} d\mathbf{x} \geq \int q'(\mathbf{x}) \log e^{\sum_{i=1}^n \lambda_i r_i(\mathbf{x})} d\mathbf{x} \\ &= \sum_{i=1}^N \lambda_i \int q'(\mathbf{x}) \cdot r_i(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^N \lambda_i a_i = \sum_{i=1}^N \lambda_i \int q^*(\mathbf{x}) \cdot r_i(\mathbf{x}) d\mathbf{x} \\ &= \int q^*(\mathbf{x}) \log e^{\sum_{i=1}^N \lambda_i r_i(\mathbf{x})} d\mathbf{x} = \int q^*(\mathbf{x}) \log \frac{q^*(\mathbf{x})}{q(\mathbf{x})} d\mathbf{x} = H(Q^* \| Q), \end{aligned}$$

where the first inequality is an equality if and only if $Q^* = Q'$, by the properties of the relative entropy functional. So Q^* is the unique minimizer. \square

Below we will calculate the relative entropy of one bivariate with respect to another. We recall that a measure is a *bivariate* with marginals $P_1 = \mathcal{N}(\mu_1, \sigma_1^2)$, $P_2 = \mathcal{N}(\mu_2, \sigma_2^2)$, and “correlation” θ , if it has Lebesgue density

$$\nu(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\theta^2}} \exp\left(-\frac{1}{2(1-\theta^2)} \left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\theta(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} \right]\right);$$

we then write $\mathcal{N}(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \theta)$ for this measure. To every such bivariate we associate

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \theta\sigma_1\sigma_2 \\ \theta\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},$$

to write more succinctly, with $\mathbf{x} = (x_1, x_2)$,

$$\nu(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det|\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1}(\mathbf{x}-\mu)\right),$$

where the first matrix is known as the *covariance matrix* of the bivariate. For two bivariate Q, Q' , the relative entropy can be expressed⁹ as

$$H(Q' \| Q) = \frac{1}{2} \left[\text{tr}(\Sigma^{-1}\Sigma') + (\mu' - \mu)^T \Sigma^{-1}(\mu' - \mu) - \log\left(\frac{\det \Sigma'}{\det \Sigma}\right) - 2 \right]. \quad (6)$$

Theorem 4.2: Let Q a bivariate with marginals $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$ and correlation θ . Let $\Pi(\mathcal{N}(\mu_1^*, \sigma_1^{*2}), \mathcal{N}(\mu_2^*, \sigma_2^{*2}))$ be the set of all Borel measures with marginals $\mathcal{N}(\mu_i^*, \sigma_i^{*2})$, for $i = 1, 2$. Then the relative entropy functional $H(\cdot \| Q)$ has a unique minimizer in $\Pi(\mathcal{N}(\mu_1^*, \sigma_1^{*2}), \mathcal{N}(\mu_2^*, \sigma_2^{*2}))$, namely the bivariate Q^* with correlation θ^* given by

$$\frac{\theta^*}{1-\theta^{*2}} = \frac{\theta}{1-\theta^2} \cdot \frac{\sigma_1^*\sigma_2^*}{\sigma_1\sigma_2}. \quad (7)$$

Proof: We actually prove a stronger statement, namely that Q^* is the minimizer of $H(\cdot \| Q)$ over \mathcal{Q}' , which is defined as the set of all Q satisfying

$$\begin{aligned} \int q'(x, y) dx dy &= 1, \\ \int x q'(x, y) dx dy &= \mu_1^*, & \int y q'(x, y) dx dy &= \mu_2^*, \\ \int x^2 q'(x, y) dx dy &= \sigma_1^{*2} + \mu_1^{*2}, & \int y^2 q'(x, y) dx dy &= \sigma_2^{*2} + \mu_2^{*2}. \end{aligned}$$

Since $Q^* \in \Pi(\mathcal{N}(\mu_1^*, \sigma_1^*), \mathcal{N}(\mu_2^*, \sigma_2^*))$, and $\Pi(\mathcal{N}(\mu_1^*, \sigma_1^*), \mathcal{N}(\mu_2^*, \sigma_2^*)) \subset \mathcal{Q}'$, it follows that Q^* is also a minimizer in $\Pi(\mathcal{N}(\mu_1^*, \sigma_1^*), \mathcal{N}(\mu_2^*, \sigma_2^*))$.

The densities q^* and q of Q^* and Q satisfy

$$\begin{aligned} \frac{q^*}{q} &= \frac{\frac{1}{2\pi\sigma_1^*\sigma_2^*\sqrt{1-\theta^{*2}}} \exp\left(-\frac{1}{2(1-\theta^{*2})} \left[\frac{(x-\mu_1^*)^2}{\sigma_1^{*2}} + \frac{(y-\mu_2^*)^2}{\sigma_2^{*2}} - \frac{2\theta^*(x-\mu_1^*)(y-\mu_2^*)}{\sigma_1^*\sigma_2^*}\right]\right)}{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\theta^2}} \exp\left(-\frac{1}{2(1-\theta^2)} \left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\theta(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right]\right)} \\ &= \frac{2\pi\sigma_1\sigma_2\sqrt{1-\theta^2}}{2\pi\sigma_1^*\sigma_2^*\sqrt{1-\theta^{*2}}} \exp\left(\frac{1}{2(1-\theta^2)} \left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\theta(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right] \right. \\ &\quad \left. - \frac{1}{2(1-\theta^{*2})} \left[\frac{(x-\mu_1^*)^2}{\sigma_1^{*2}} + \frac{(y-\mu_2^*)^2}{\sigma_2^{*2}} - \frac{2\theta^*(x-\mu_1^*)(y-\mu_2^*)}{\sigma_1^*\sigma_2^*}\right] \right) \\ &= \frac{\sigma_1\sigma_2\sqrt{1-\theta^2}}{\sigma_1^*\sigma_2^*\sqrt{1-\theta^{*2}}} \exp\left(\left[\frac{1}{2\sigma_1^2(1-\theta^2)} - \frac{1}{2\sigma_1^{*2}(1-\theta^{*2})}\right]x^2 \right. \\ &\quad \left. + \left[\frac{1}{2\sigma_2^2(1-\theta^2)} - \frac{1}{2\sigma_2^{*2}(1-\theta^{*2})}\right]y^2 + \left[\frac{\theta^*}{(1-\theta^{*2})\sigma_1^*\sigma_2^*} - \frac{\theta}{(1-\theta^2)\sigma_1\sigma_2}\right]xy \right. \\ &\quad \left. + \left[\frac{\mu_1}{(1-\theta^2)\sigma_1^2} - \frac{\theta\mu_2}{(1-\theta^2)\sigma_1\sigma_2} + \frac{\mu_1^*}{(1-\theta^{*2})\sigma_1^{*2}} - \frac{\theta^*\mu_2^*}{(1-\theta^{*2})\sigma_1^*\sigma_2^*}\right]x \right. \\ &\quad \left. - \left[\frac{\mu_1}{(1-\theta^2)\sigma_1^2} - \frac{\theta\mu_2}{(1-\theta^2)\sigma_1\sigma_2} + \frac{\mu_1^*}{(1-\theta^{*2})\sigma_1^{*2}} - \frac{\theta^*\mu_2^*}{(1-\theta^{*2})\sigma_1^*\sigma_2^*}\right]y \right. \\ &\quad \left. + \left[\frac{\theta\mu_1\mu_2}{\sigma_1\sigma_2(1-\theta^2)} - \frac{\theta^*\mu_1^*\mu_2^*}{\sigma_1^*\sigma_2^*(1-\theta^{*2})}\right] \right) \\ &\stackrel{(7)}{=} \exp(Ax^2 + By^2 + Cx + Dy + E), \end{aligned}$$

for suitable constants A, B, C, D , and E . Thus Q^* has a density of the form $q^*(\mathbf{x}) = q(\mathbf{x}) \exp(\sum_{i=1}^n \lambda_i r_i(\mathbf{x}))$. Therefore by Theorem IV.1, Q^* minimizes the functional $H(\cdot \| Q)$ over $\Pi(\mathcal{N}(\mu_1^*, \sigma_1^*), \mathcal{N}(\mu_2^*, \sigma_2^*))$. \square

V. ASYMPTOTIC BEHAVIOUR OF THE RATE FUNCTIONAL

Theorem 5.1: Let $\mathcal{N}(\mu_0, \sigma_0^2)$ and $\mathcal{N}(\mu, \sigma^2)$ be two normal distributions. Then the rate functional $J_h(\mathcal{N}(\mu, \sigma^2); \mathcal{N}(\mu_0, \sigma_0^2))$ is given by

$$\frac{1}{2} \left[\frac{(\sigma - \sigma_0)^2 + (\mu - \mu_0)^2}{2h} + \frac{1 - \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma}}{\frac{h}{\sigma \sigma_0}} - \log \left(\sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma} \right) - \log \frac{\sigma}{\sigma_0} - 1 \right]. \quad (8)$$

Proof: We have

$$J_h(\mathcal{N}(\mu, \sigma^2); \mathcal{N}(\mu_0, \sigma_0^2)) = \inf_{Q \in \Gamma(\mathcal{N}(\mu_0, \sigma_0^2), \mathcal{N}(\mu, \sigma^2))} H(Q \| Q_{0 \rightarrow h}),$$

where $Q_{0 \rightarrow h} = \mathcal{N}(\mu_0, \sigma_0^2) * P_h$. The probability distribution of $Q_{0 \rightarrow h}$ is given by

$$q_{0 \rightarrow h} = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right) \cdot \frac{1}{\sqrt{4\pi}h} \exp\left(-\frac{(x-y)^2}{4h}\right).$$

It is easy to see that $Q_{0 \rightarrow h}$ is then the Bivariate

$$\mathcal{N}(\mu_0, \sigma_0^2, \mu_0, \sigma_h^2, \theta) \quad \text{where} \quad \sigma_h^2 = \sigma_0^2 + 2h \quad \text{and} \quad \theta = \frac{\sigma_0}{\sigma_h}; \quad (9)$$

for the corresponding matrices

$$\Sigma_{0 \rightarrow h} = \begin{pmatrix} \sigma_0^2 & \theta \sigma_0 \sigma_h \\ \theta \sigma_0 \sigma_h & \sigma_h^2 \end{pmatrix}, \quad \mu_{0 \rightarrow h} = \begin{pmatrix} \mu_0 \\ \mu_0 \end{pmatrix},$$

we have trivially

$$\det(\Sigma_{0 \rightarrow h}) = \sigma_h^2 \sigma_0^2 - \theta^2 \sigma_0^2 \sigma_h^2 = (1 - \theta^2) \sigma_0^2 \sigma_h^2 = 2h \sigma_0^2$$

and

$$\Sigma_{0 \rightarrow h}^{-1} = \frac{1}{2h \sigma_0^2} \begin{pmatrix} \sigma_h^2 & -\theta \sigma_0 \sigma_h \\ -\theta \sigma_0 \sigma_h & \sigma_0^2 \end{pmatrix}.$$

By Theorem 4.2, it follows that the minimizer of $H(\cdot \| Q_{0 \rightarrow h})$ in $\Pi(\mathcal{N}(\mu_0, \sigma_0), \mathcal{N}(\mu, \sigma))$ is $Q^* = \mathcal{N}(\mu_0, \sigma_0^2, \mu, \sigma^2, \theta^*)$, with corresponding matrices

$$\Sigma^* = \begin{pmatrix} \sigma_0^2 & \theta^* \sigma_0 \sigma \\ \theta^* \sigma_0 \sigma & \sigma^2 \end{pmatrix}, \quad \mu^* = \begin{pmatrix} \mu_0 \\ \mu \end{pmatrix},$$

where θ^* satisfies

$$\frac{\theta^*}{1 - \theta^{*2}} = \frac{\theta}{1 - \theta^2} \cdot \frac{\sigma_0 \sigma}{\sigma_0 \sigma_h} = \frac{\sigma_0 \sigma_h}{2h} \cdot \frac{\sigma_0 \sigma}{\sigma_0 \sigma_h} = \frac{\sigma_0 \sigma}{2h}.$$

By solving the above quadratic equation, we find

$$\theta^* = \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma}. \quad (10)$$

So for the rate functional as in (6),

$$H(Q^* \| Q_{0 \rightarrow h}) = \frac{1}{2} \left[\text{tr}(\Sigma_{0 \rightarrow h}^{-1} \Sigma^*) + (\mu^* - \mu)^T \Sigma_{0 \rightarrow h}^{-1} (\mu^* - \mu) - \log \left(\frac{\det \Sigma^*}{\det \Sigma_{0 \rightarrow h}} \right) - 2 \right],$$

we have

$$\begin{aligned} \text{tr}(\Sigma_{0 \rightarrow h}^{-1} \Sigma^*) &= \text{tr} \left(\frac{1}{2h \sigma_0^2} \begin{pmatrix} \sigma_h^2 & -\theta \sigma_0 \sigma_h \\ -\theta \sigma_0 \sigma_h & \sigma_0^2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_0^2 & \theta^* \sigma_0 \sigma \\ \theta^* \sigma_0 \sigma & \sigma^2 \end{pmatrix} \right) \\ &= \frac{\sigma_h^2 \sigma_0^2 - 2\theta \theta^* \sigma_0^2 \sigma \sigma_h + \sigma_0^2 \sigma^2}{2h \sigma_0^2} = \frac{\sigma_h^2 - 2\theta \theta^* \sigma \sigma_h + \sigma^2}{2h} \\ &\stackrel{(9)}{=} \frac{\sigma_0^2 + 2h - 2\theta \theta^* \sigma \sigma_h + \sigma^2}{2h} \\ &= \frac{(\sigma - \sigma_0)^2}{2h} + \frac{2\sigma \sigma_0 - 2\theta \theta^* \sigma \sigma_h}{2h} + 1 \\ &\stackrel{(9)}{=} \frac{(\sigma - \sigma_0)^2}{2h} + \frac{1 + \frac{h}{\sigma_0 \sigma} - \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1}}{\frac{h}{\sigma \sigma_0}} + 1. \end{aligned} \quad (11)$$

Also,

$$\begin{aligned} (\mu^* - \mu)^T \Sigma_{0 \rightarrow h}^{-1} (\mu^* - \mu) &= \begin{pmatrix} 0 \\ \mu - \mu_0 \end{pmatrix}^T \frac{1}{2h \sigma_0^2} \begin{pmatrix} \sigma_h^2 & -\theta \sigma_0 \sigma_h \\ -\theta \sigma_0 \sigma_h & \sigma_0^2 \end{pmatrix} \begin{pmatrix} 0 \\ \mu - \mu_0 \end{pmatrix} \\ &= \frac{(\mu - \mu_0)^2 \sigma_0^2}{2h \sigma_0^2} = \frac{(\mu - \mu_0)^2}{2h}. \end{aligned} \quad (12)$$

Finally,

$$\begin{aligned}
 \log \left(\frac{\det \Sigma^*}{\det \Sigma_{0 \rightarrow h}} \right) &= \log \left(\frac{(1 - \theta^{*2}) \sigma_0^2 \sigma^2}{(1 - \theta^2) \sigma_0^2 \sigma_h^2} \right) \stackrel{(7)}{=} \log \frac{\theta^* \sigma_0^2 \sigma^2 \sigma_0 \sigma_h}{\theta \sigma_0^2 \sigma_h^2 \sigma_0 \sigma} = \log \frac{\theta^*}{\theta} + \log \frac{\sigma}{\sigma_h} \\
 &\stackrel{(9)}{\stackrel{(10)}}{=} \log \left(\sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma} \right) - \log \frac{\sigma_0}{\sigma_h} + \log \frac{\sigma}{\sigma_h} \\
 &= \log \left(\sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma} \right) + \log \frac{\sigma}{\sigma_0}. \tag{13}
 \end{aligned}$$

So by (11), (12), and (13),

$$\begin{aligned}
 H(Q^* \| Q_{0 \rightarrow h}) &= \\
 \frac{1}{2} \left[\frac{(\sigma - \sigma_0)^2 + (\mu - \mu_0)^2}{2h} + \frac{1 - \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma}}{\frac{h}{\sigma \sigma_0}} - \log \left(\sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma} \right) - \log \frac{\sigma}{\sigma_0} - 1 \right],
 \end{aligned}$$

as claimed. \square

We continue with the proof of the central statement, Theorem 3.2. We start with two observations. First, for Gaussian measures, the weak topology is equivalent to the one induced by the Wasserstein metric [Ref. 11, Theorem 7.12]. Second, by Ref. 10, for two Gaussians $\mathcal{N}(\mu_0, \sigma_0^2)$ and $\mathcal{N}(\mu, \sigma^2)$, it holds that

$$W^2(\mathcal{N}(\mu_0, \sigma_0^2), \mathcal{N}(\mu, \sigma^2)) = (\mu - \mu_0)^2 + (\sigma - \sigma_0)^2. \tag{14}$$

And so

$$\begin{aligned}
 &\left[J_h(\mathcal{N}(\mu, \sigma^2); \mathcal{N}(\mu_0, \sigma_0^2)) - \frac{W_2^2(\mathcal{N}(\mu, \sigma^2), \mathcal{N}(\mu_0, \sigma_0^2))}{4h} - \frac{1}{2} E(\mathcal{N}(\mu, \sigma^2)) + \frac{1}{2} E(\mathcal{N}(\mu_0, \sigma_0^2)) \right] \\
 &\leq \frac{1}{2} \left| \frac{1 + \frac{h}{\sigma_0 \sigma} - \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1}}{\frac{h}{\sigma \sigma_0}} - \log \left(\sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma} \right) - 1 \right| \\
 &\leq \left| \frac{1 + \frac{h}{\sigma_0 \sigma} - \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1}}{\frac{2h}{\sigma \sigma_0}} - \frac{1}{2} \right| + \frac{1}{2} \left| \log \left(\sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma} \right) \right| \\
 &\leq \left| \frac{1 - \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1}}{\frac{2h}{\sigma \sigma_0}} \right| + \frac{1}{2} \left| \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} - \frac{h}{\sigma_0 \sigma} - 1 \right| \\
 &\leq \left| \frac{1 - \sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1}}{\frac{2h}{\sigma \sigma_0}} \right| + \left| \frac{1}{2} \frac{2 \frac{h}{\sigma \sigma_0}}{\sqrt{\frac{h^2}{\sigma_0^2 \sigma^2} + 1} + \frac{h}{\sigma_0 \sigma} + 1} \right| \\
 &\leq \frac{o(h)}{\sigma}. \tag{15}
 \end{aligned}$$

Now for fixed $\delta > 0$, we have uniform convergence for $\sigma' > \delta$. For the Γ -convergence, we get the lower bound part by the local uniform convergence and the lower semi-continuity of the limit. As a recovery sequence, we can choose the fixed sequence $P_h = \mathcal{N}(\mu, \sigma^2)$. \square

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